

Linear Transformations

Definition: Let V and W be vector spaces. A mapping $L:V \rightarrow W$ is a **linear transformation** if

- (a) $L(v_1 + v_2) = L(v_1) + L(v_2)$ for all $v_1, v_2 \in V$, and
- (b) $L(\alpha v) = \alpha L(v)$ for all $v \in V$ and scalar α .

The two parts of this definition are sometimes combined into one condition:

$$L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2).$$

Also, a word should be said about the vector addition. There is no guarantee that the two different vector spaces (V and W) have the same vector addition. So in part (a) above, be careful to remember that the addition on the left side of the equation takes place in V and the addition on the right takes place in W .

$$\begin{array}{ccc} L(v_1 + v_2) = L(v_1) + L(v_2) \\ \swarrow \quad \quad \quad \nwarrow \\ \text{Addition is in } V \quad \quad \quad \text{Addition is in } W \end{array}$$

Example 1: Define $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $L(x) = 3x$. Verify that this is a linear transformation.

$$L(x + y) = 3(x + y) = 3x + 3y = L(x) + L(y), \text{ and}$$

$$L(\alpha x) = 3(\alpha x) = \alpha(3x) = \alpha L(x)$$

Clearly, this can be generalized to the fact that $L(x) = kx$ for any k is a linear transformation. If k is positive, we can think of this mapping as a stretching or shrinking by a factor of k .

Example 2: Define $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$.

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} y_1 \\ 0 \end{bmatrix} = L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + L\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right), \text{ and}$$

$$L\left(\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}\right) = \begin{bmatrix} \alpha x_1 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \alpha L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

This linear transformation can be thought of as a projection onto the x_1 -axis.

Example 3: Define $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$.

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + y_1 \\ -x_2 - y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ -y_2 \end{bmatrix} = L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + L\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right)$$

$$L\left(\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}\right) = \begin{bmatrix} \alpha x_1 \\ -\alpha x_2 \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} = \alpha L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

This linear transformation has the effect of reflecting vectors about the x_1 -axis.

Example 4: Define $L: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ by $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_3 \\ x_3 - x_2 \end{bmatrix}$.

$$\begin{aligned} L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) &= L\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}\right) = \begin{bmatrix} (x_1 + y_1) + (x_3 + y_3) \\ (x_3 + y_3) - (x_2 + y_2) \end{bmatrix} = \begin{bmatrix} (x_1 + x_3) + (y_1 + y_3) \\ (x_3 - x_2) + (y_3 - y_2) \end{bmatrix} \\ &= \begin{bmatrix} x_1 + x_3 \\ x_3 - x_2 \end{bmatrix} + \begin{bmatrix} y_1 + y_3 \\ y_3 - y_2 \end{bmatrix} = L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + L\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) \end{aligned}$$

$$L\left(\alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = L\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}\right) = \begin{bmatrix} \alpha x_1 + \alpha x_3 \\ \alpha x_3 - \alpha x_2 \end{bmatrix} = \begin{bmatrix} \alpha(x_1 + x_3) \\ \alpha(x_3 - x_2) \end{bmatrix} = \alpha \begin{bmatrix} x_1 + x_3 \\ x_3 - x_2 \end{bmatrix} = \alpha L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$$

Example 5: Define $L: \mathbb{R}^2 \rightarrow \mathbb{R}$ by $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_2 + 2x_1$.

$$\begin{aligned} L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) &= L\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}\right) = (x_2 + y_2) + 2(x_1 + y_1) = (x_2 + 2x_1) + (y_2 + 2y_1) \\ &= L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + L\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) \end{aligned}$$

$$L\left(\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}\right) = \alpha x_2 + 2(\alpha x_1) = \alpha(x_2 + 2x_1) = \alpha L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

Example 6: Let A be an $m \times n$ matrix. Define $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ by $L(x) = Ax$.

$$L(x + y) = A(x + y) = Ax + Ay = L(x) + L(y)$$

$$L(\alpha x) = A(\alpha x) = \alpha(Ax) = \alpha L(x)$$

Ok, so is every mapping a linear transformation? Of course not.

Example 7: Define $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ x_2 \end{bmatrix}$.

$$L\left(\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ \alpha x_2 \end{bmatrix} \neq \alpha \begin{bmatrix} 5 \\ x_2 \end{bmatrix}$$

Example 8: Define $L: \mathbb{R} \rightarrow \mathbb{R}$ by $L(x) = x^2$.

$$L(x + y) = (x + y)^2 = x^2 + 2xy + y^2 \neq L(x) + L(y)$$

Example 9: Define $L : P_5 \rightarrow \mathbb{R}$ by $L(f) = \deg(f)$.

Since $\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$, $\deg(f + g) \neq \deg(f) + \deg(g)$.

Theorem 1: Let $L : V \rightarrow W$ be a linear transformation. Then $L(0_V) = 0_W$.

Proof: This follows from condition (b) in the definition of linear transformation if we let $\alpha = 0$.

Definition: Let $L : V \rightarrow W$ be a linear transformation. The *kernel* of L is the set $\ker(L) = \{v \in V : L(v) = 0_W\}$. By the previous theorem, this set is always nonempty.

Theorem 2: Let $L : V \rightarrow W$ be a linear transformation. Then the $\ker(L)$ is a subspace of V .

Proof: Let $x, y \in \ker(L)$. Then $L(x + y) = L(x) + L(y) = 0 + 0 = 0$, so $(x + y) \in \ker(L)$ also. Similarly, $L(\alpha x) = \alpha L(x) = \alpha 0 = 0$, so $\alpha x \in \ker(L)$.

Definition: Let $L : V \rightarrow W$ be a linear transformation and let S be a subset of V . The *image* of S is the set $L(S) = \{w \in W : w = L(v) \text{ for some } v \in S\}$. The *range* is the set $L(V)$.

Theorem 3: Let $L : V \rightarrow W$ be a linear transformation. If S is a subspace of V , then $L(S)$ is a subspace of W .

Proof: Since $0_V \in S$ and $L(0_V) = 0_W$, $0_W \in L(S)$ so $L(S)$ is nonempty. Let $x, y \in L(S)$. Then $x = L(v_1)$ and $y = L(v_2)$ for some $v_1, v_2 \in S$. So $x + y = L(v_1) + L(v_2) = L(v_1 + v_2)$. Since S is a subspace of V , $v_1 + v_2 \in S$. So $(x + y) \in L(S)$. Likewise, $\alpha x = \alpha L(v_1) = L(\alpha v_1)$, so $\alpha x \in L(S)$.

Example 10: Find the kernel of each linear transformation.

(a) L defined in Example 2.

(b) L defined in Example 3.

(c) $D : P_4 \rightarrow P_4$ defined by $D(f) = f'$ (D is usually called the

differentiation operator for obvious reasons).

(a) $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$. For this to be $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, x_1 must be zero. So the $\ker(L) = \{(0, x_2)^T : x_2 \in \mathbb{R}\}$.

(b) $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$. For this to be $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $x_1 = x_2 = 0$. So the $\ker(L) = \{(0, 0)\}$.

(c) The zero element in P_4 is 0. The only polynomials with derivative zero are the constant polynomials (i.e. polynomials with degree 0). So $\ker(D) = P_1$.

Homework

1. Let a be a fixed vector in \mathbb{R}^2 . The mapping $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $L(x) = x + a$ is called a *translation*. Under what conditions is this a linear transformation?

2. Determine whether the following mappings are linear transformations from \mathbb{R}^2 to \mathbb{R}^3 .

(a) $L(x) = (x_1, x_2, 1)^T$

(b) $L(x) = (x_1, x_2, x_1 + x_2)^T$

(c) $L(x) = (x_1, x_2, 0)^T$

(d) $L(x) = (x_1, x_2, x_1^2 + x_2^2)^T$

3. Determine the kernel of each linear transformation $L: P_3 \rightarrow P_3$.

(a) $L(p(x)) = xp'(x)$

(b) $L(p(x)) = p(x) - p'(x)$

(c) $L(p(x)) = p(0)x + p(1)$

4. A linear transformation is called *one-to-one* (or *injective*) if $L(v_1) = L(v_2)$ implies $v_1 = v_2$. Show that L is one-to-one if and only if $\ker(L) = \{0\}$.

5. Let $L: V \rightarrow W$ be a linear transformation. The image $L(V)$ is always a subspace of W . A linear transformation is called **onto** (or **surjective**) if $L(V) = W$. Let D be the differentiation operator and let $S = \{p \in P_3 : p(0) = 0\}$. Show that:

- (a) D maps P_3 onto P_2 but is not one-to-one.
- (b) $D: S \rightarrow P_3$ is one-to-one, but is not onto.